NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

A GAME THEORY APPROACH TO SEARCH

by

Clark Wallis Pritchett

October 1982

Thesis Advisor: G. F. Lindsay

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A124631	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Game Theory Approach to Search		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; October 1982
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Clark Wallis Pritchett		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE October 1982
		13. NUMBER OF PAGES 69
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Search   Detection

Game Theory

Area Patrol

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A model of search of an area is developed using the probability of detection as the measure of effectiveness. The area is partitioned into two pieces. Two search units, with different capabilities, attempt to detect one evader. If the search efforts of the two units do not overlap, the probability of detection is the same no matter how the area is partitioned.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

#20 - ABSTRACT - (CONTINUED)

An approach based upon Game theory is also developed. The variance of the probability of detection is computed and used to select strategies for searching.

Approved for public release; distribution unlimited.

A Game Theory Approach to Search

by

Clark Wallis Pritchett
B.S., George Washington University, 1965

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL

October 1982

Author: Clark W. Pritchett

Approved by: ______________________ Thesis Advisor

______________________ Second Reader

____
Chairman, Department of Operations Research

W. M. Woods
Dean of Information and Policy Sciences

ABSTRACT

A model of search of an area is developed using the probability of detection as the measure of effectiveness. The area is partitioned into two pieces. Two search units, with different capabilities, attempt to detect one evader. If the search efforts of the two units do not overlap, the probability of detection is the same no matter how the area is partitioned. An approach based upon Game theory is also developed. The variance of the probability of detection is computed and used to select strategies for searching.

TABLE OF CONTENTS

LIST OF TABLES

Table #

LIST OF FIGURES

Figure #

I. INTRODUCTION

Throughout history, searching, on one form or another has been one of man's constant endeavors. Prehistoric man hunted for his food, both plant and animal. The more successful searchers understood the characteristics of their quary and search area. They could then eliminate some areas from consideration and concentrate on others. For example, if certain plants only grew in the shade, open fields would be eliminated from the search area.

In searches involving people, this characteristic information can be exploited by both the searcher and his quary (the evader). If the evader knows the tendencies of the searcher, he can take the appropriate actions to reduce his chances of being found. The knowledge of an adversary's action (intelligence in military terms) is always sought to give one side an advantage. Certainly the submarine commander wants to know where the convoy will transit, just as much as the convoy commander wants to know where the submarine will hunt [Ref. 1].

We can see that to fall into patterns of searching (or evading) is analogous to providing the other party with information that he can exploit. Since a pattern or determinant strategy provides information to the adversary, it is only logical not to be predictable. Therefore, the searcher and evader should be unpredictable, or random, in their choice

of search actions. Search patterns that are random do not yield repeatable results from each trial but certain quantities such as the probability of detection are amenable to mathematical calculation.

Each time the searcher faces a choice with a known set of conditions, he should choose his action randomly to avoid falling into a pattern. If the search action was repeatable, the same choice would be repeated each time and hence a pattern would develo

A prime quantity of interest in a search is the probability of detection (POD). The probability of detection is a measure of effectiveness (MOE) of the search. The basic problem is to compute this MOE. The next step is usually to optimize the probability of detection. Optimize means maximize to the searcher and minimize to the evader. Occasionally other pieces of information are required such as the probability of detection under an extreme, or most probable, set of conditions.

This paper will explore the problem of one or more searching units searching for a single evader. It is assumed that neither party has tendencies that may be exploited by the other to predict his actions. The actions of the searcher and evader are random in that act ons or choices are not predictable.

A border search problem will be developed initially. The probability of detection is defined and a cookie cutter model is introduced. The searcher has two search units with

different capabilities attempting to detect a single evader. Following this, a more complex problem is addressed. Two search units of different capabilities attempt to detect a single evader in an area which is partitioned into two pieces.

The area search problem is then addressed as a two person zero sum game. Each entry in the payoff matrix is the probability of the searcher detecting the evader for the strategies picked by each player. The optimal probabilities of playing each strategy are computed. The expected value results of the game indicate that all pairs of search strategies yield the same expected value of the game. The choice of strategies is made based upon minimum variance. The search area is then partitioned into three pieces and an example is presented.

The concept of random search is next introduced. The evader is allowed the latitude of moving within the operating areas. The problem is modeled as a game and the expectation and variance are computed. In the case of random search, the pair of strategies of the searcher which yield the largest expected value of the game would be chosen for the search.

II. THE NATURE OF THE SEARCH PROBLEM

Search units are generally constrained by range, speed, endurance or other restrictions such as hours of daylight available. The time and speed constraints can be combined into an effective range[1] of the searching unit. If there are two units available to search a region, it is logical to want to find the best way to partition the region to optimize the search. Two partitioning problems will be addressed in this section. The first is the search of a border. The second problem is that of searching an area. Under certain conditions and constraints, the area searching problem can be reduced to the same mathematical form as the border, or line searching problem.

A. BORDER SEARCHING PROBLEM

We consider the problem of insurgents or smugglers attempting to cross a border such as a river or an international boundary. The border must not necessarily be straight but we require that it does not have extreme curvature or corners. We define the person, or unit, attempting to cross the border, as the *evader*. Let the patrolling person, or unit, be defined as the *searcher*. We will assume that the border is very long

---

[1] Range = Average Speed X Endurance.

so that at any time the searcher cannot observe the entire length of the border.[1]

The objective of the evader is to cross the border without being detected by the searcher. The searcher's objective is to detect the evader when he attempts to cross the border. A detection can only occur when the evader appears in the field of view of the searcher's sensor. The sensors could be acoustic (sonar), electromagnetic (radar), or optical (infrared or image enhancing devices). Of course, the human eye is the sensor we are all most familiar with.

The searcher and the evader have diametrically opposed views of success and failure. The searcher considers the detection of the evader a success while the evader calls it a failure. If the evader is not detected, he considers that a success, while the searcher considers it to be a failure. The searcher and evader can both observe the same set of events and reach opposite conclusions about success and failure.

Consider the following model of the border search problem in Figure 1. This model is applicable while the searcher is actively on station searching for evaders crossing the border. The evader is attempting to cross the border into friendly territory. The probability of detection is 1.0 if the evader passes within the sweep width W, otherwise it is 0.0. It is

---

[1]The towers of the great wall of China were constructed so the entire length of the border could be viewed simultaneously by many observers.

W = Sweepwidth

V = Searcher Speed

U = Evader Speed

S = Border Length

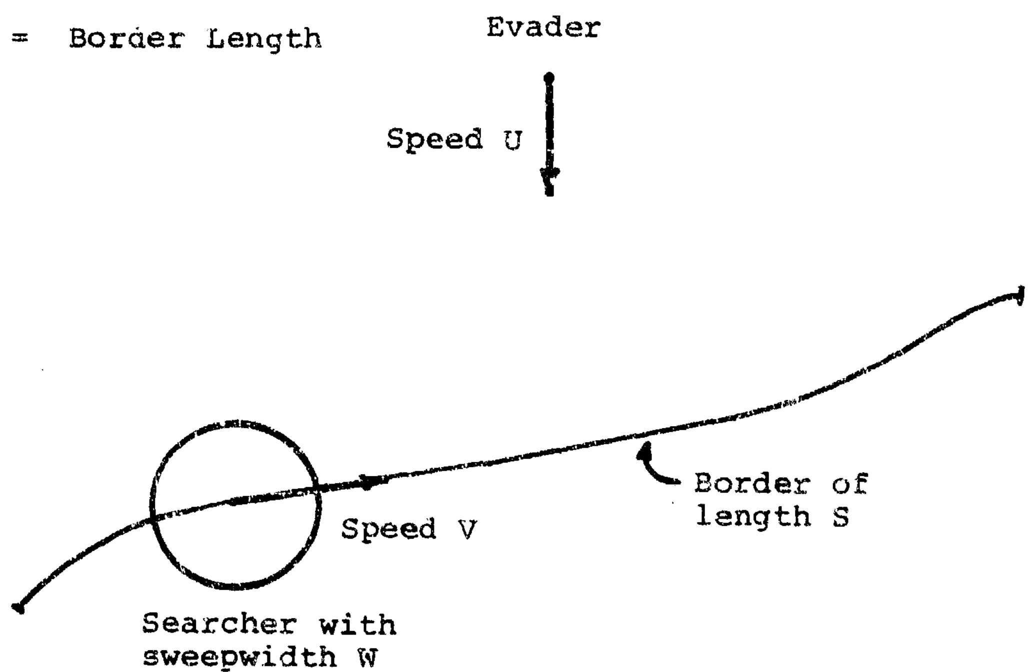


Figure 1. Model of Border Search

assumed that the border length S is much greater than the sweepwidth W. The searcher travels at speed V along the border. The evader is equally likely to cross anywhere along the length of the border. The searcher traverses the border in one direction for the duration of his endurance.

It is necessary to state some further assumptions and consequences of the model. We assume that the searcher has a much greater sweepwidth than the counterdetection range of the evader. Therefore, no evasive action may be taken by the evader. In addition there can be no missed detections or false alarms. The evader can only be detected when he falls within the sweepwidth of the searcher's sensors.

The evader approaches the border at speed U, and at an angle, θ. The searcher travels at speed V along the border. We choose an orthogonal coordinate system, perpendicular (⊥) and parallel (//) to the border (see Figure 2) and resolve the speed of the searcher and evader into components in this coordinate system.

When the evader crosses the border, he must travel the distance L = W/cos θ, in which he is subject to detection. The evader traverses L in time t, where:

$$t = \frac{L}{U} = \frac{\frac{W}{\cos\theta}}{U} = \frac{\frac{W}{U_\perp/U}}{U} = \frac{W}{U_\perp} \qquad (1)$$

We see in Equation (1) that only the velocity component of the evader perpendicular to the border enters into the

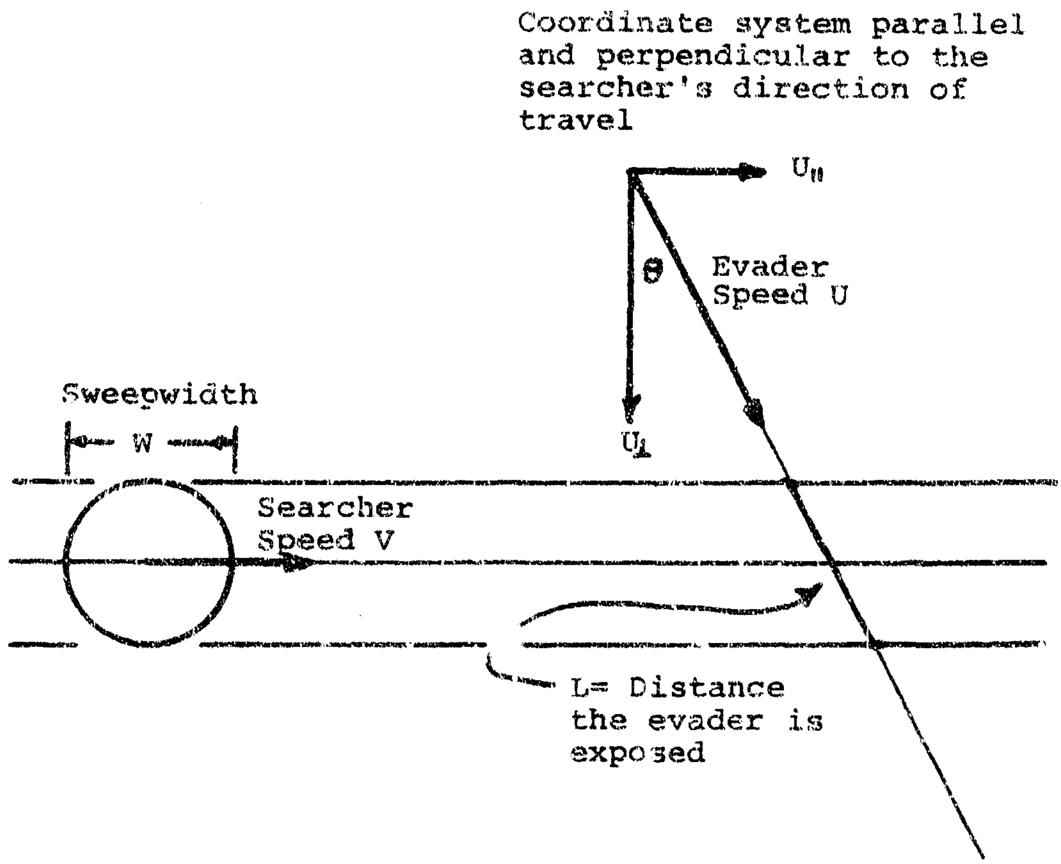


Figure 2. Evader Attempting to Cross Border

time to traverse the sweep width band. It is apparent that the evader should not be exposed to possible detection any longer than necessary. The evader does not know the searcher's direction of travel along S. The speed component U should be as large as possible to minimize the time to cross the border. This occurs when the evader crosses perpendicular to the border. Under these conditions $\theta = 0$, $U_{\perp} = U$ and $U_{\parallel} = 0$. In all subsequent developments, the evader will cross the border in the most advantageous manner i.e., perpendicular to the border.

The searcher is not allowed to cover a portion of the border that he has already traversed, and only travels in one direction. However, he may come to the end of the border before he has exhausted his search time. We can prevent this from happening by requiring that the searcher always begin his search at least $VT + W/2$ from either end of the border. If he moves toward the end of the border for time T at speed V, he is still $W/2$ away from the end when he finishes the search, where $W/2$ is the search radius.

The measure of effectiveness in this problem has been chosen to be the probability of detecting the evader. It may be possible to determine the probability of detection experimentally. The initial conditions of a search would be determined for the searcher and the evader. The evader would select from a uniform distribution, a point where to attempt to cross the border. The searcher would select from

a uniform distribution, where to start his search within the interval VT + W/2 from each end of the border. Then with a flip of a fair coin, he would decide which direction in which to traverse the border. A series of these trials would be conducted and the number of detections would be recorded. The probability of detection would be estimated from the ratio of the number of detections to the number of trials. The above procedure could be accomplished more easily in a simulation than in a physical experiment. A more appealing alternative is to compute the probability of detection directly.

For a given evader's speed U, the effectiveness of the searcher is a function of his speed V, the sensor sweepwidth W and the border length S. The probability of detection is only defined if the evader is present. The search speed, sweepwidth and length of border alone are not enough to define the probability of detection, though some information is available by taking the ratio of sweepwidth and border length. This coverage factor, W/S, is one measure of the searcher's capability.

If the searcher could traverse the entire border during the increment of time that the evader was attempting the crossing, the evader would surely be detected. If the searcher were stationary, the evader would certainly not be detected unless he crossed the border in the field of view of the stationary sensor of the searcher. In the first case, the probability of detection is 1.0 and in the second case,

it is equal to the coverage factor, W/S. In the problem under consideration, some part of the total length of the border is traversed by the searcher during the time that it takes the evader to make his crossing. The probability of detection is defined to be the ratio of the length of the border searched, R (in the time interval, t, that it takes the evader to cross the border), to the length of the border, S. This is the probability of detection given the evader is crossing the border S, i.e., POD|S. Here,

$$\mathrm{POD}|S = \frac{R}{S} = \frac{W + V^*t}{S} = \frac{W + V^*W/U}{S}, \tag{2}$$

where $U > 0$, $V \geq 0$.

Rearranging terms, the following expression for the probability of detection results:

$$\mathrm{POD}|S = \frac{W}{S} \times \left(1 + \frac{V}{U}\right), \tag{3}$$

where $U > 0$, $V \geq 0$.

In Equation (3), W/S is the coverage factor. Consider the example where the searcher's speed is V = 0, such as when he occupies a watch tower on the border. If the evader crosses within his field of view, W, he is detected. Even with no speed, there still would be a finite POD. The ratio of speeds in the second term represents the increase in the probability of detection due to the speed of the searcher.

This ratio, V/U, may be interpreted as the dynamic enhancement of the static probability of detection [Ref. 2].

We will now determine the upper bound on the POD|S. Figure 3 shows the allowable interval in which the searcher may begin his search.

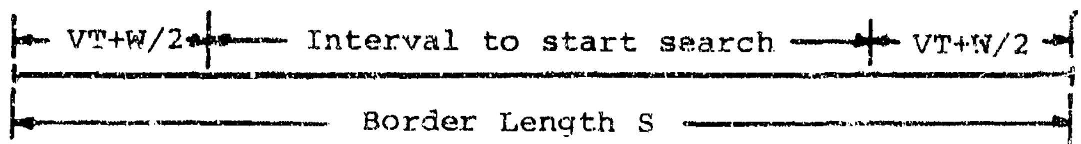


Figure 3. Interval to Start Search

During the interval t that the evader is crossing the border,

$$S > 2(vt \pm W/2). \tag{4}$$

Rearranging terms yields

$$1 > \frac{2vt}{S} + \frac{W}{S}.$$

Recall that $t = W/U$ and substitute into b, yielding

$$1 > \frac{2VW}{SU} + \frac{W}{S}, \text{ or}$$

$$1 > \frac{W}{S}(1 + 2\,\frac{V}{U}).$$

Since

$$\text{POD}|S = \frac{W}{S}(1 + \frac{V}{U}),$$

we have

$$1 > \frac{W}{S}(1 + \frac{2V}{U}) \geq \frac{W}{S}(1 + \frac{V}{U}), \tag{5}$$

and

$$\text{POD}|S < 1.$$

If we examine the probability of detection (Equations 2 and 3) we may confirm some intuitive ideas about searching. The POD is greater if the searcher can see farther (greater sweepwidth, W) or cover more ground in a given time (greater speed, V). The probability of detection is less if the evader can cross the border more quickly (greater speed, U) or operate along an expanded border (i.e., increased S).

B. TWO SEARCHING UNITS ON THE BORDER

In this section we will assume that there are two searchers, Unit 1 and Unit 2 searching for a single evader crossing the border. Each search unit has a different sweepwidth (W_1 and W_2) and speed (V_1 and V_2). The speed of the evader is U. The ranges, R_1 and R_2 that represent the search capability of

each unit while the evader is crossing the border can be easily computed. Here:

$$R_1 = W_1(1 + \frac{V_1}{U}) \quad \text{and}$$

$$R_2 = W_2(1 + \frac{V_2}{U}) \tag{6}$$

These equations are of the same form as the numerator in Equation (2).

It is reasonable to have some rationale for the division of labor between the two searching units. Suppose that we partition the border length, S, into two pieces of lengths S_1 and S_2, where $S_1 + S_2 = S$. We assign Unit 1 to the length of border S_1 and Unit 2 to length of border S_2, as shown in Figure 4.

The probabilities of detection of Unit 1 and Unit 2 (in S_1 and S_2 respectively) are:

$$POD_1|S_1 = \frac{R_1}{S_1} \quad \text{and}$$

$$POD_2|S_2 = \frac{R_2}{S_2}. \tag{7}$$

The probability of detecting the evader, P(Det), is the total probability, i.e.,

$$P(Det) = P(Det|S_1)*P(S_1) + P(Det|S_2)*P(S_2). \tag{8}$$

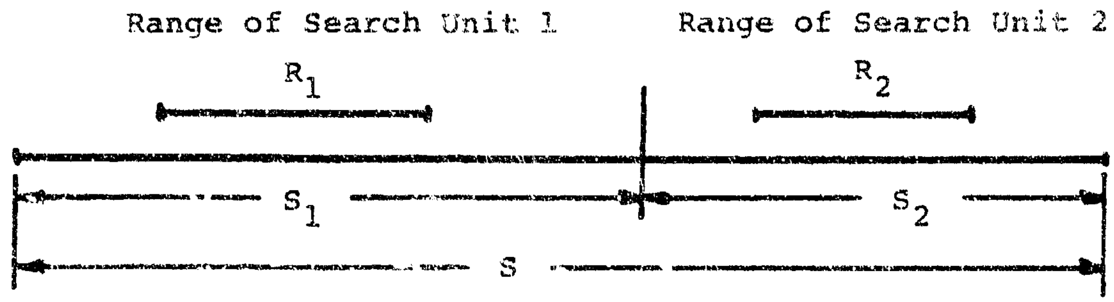


Figure 4. Border of Length S Partitioned into Two Pieces

where:

$P(S_i)$ = Probability the evader will cross the border in segment S_i, $i = 1,2$.

Since the evader is assumed to be equally likely to cross the border anywhere along its length,

$$P(S_i) = \frac{S_i}{\sum_i S_i} . \tag{9}$$

Now, reducing Equation (8) using Equations (7) and (9) gives:

$$P(\mathrm{Det}) = \frac{R_1}{S_1}\,\frac{S_1}{S_1+S_2} + \frac{R_2}{S_2}\,\frac{S_2}{S_1+S_2} = \frac{R_1+R_2}{S_1+S_2} = \frac{R}{S}, \tag{10}$$

where:

$R_1+R_2 = R$ = the total non-overlapping length of the border patrolled by the two search units.

The implicit idea in partitioning the border was to do it in such a way that the probability of detection was optimized. We see from the form of Equation (10) that the partitioning of the border into two individual segments has no effect on the overall probability of detection. This is the result of the evader being equally likely to cross the border at any point along its length, and of not allowing any overlap by the search units.

C. TWO SEARCHING UNITS IN AN AREA

The next logical extension of the searching problem is to search an area instead of a border. We will consider two search units in an area, searching for an evader who is operating at a fixed location. If the area swept by a search unit's sensor contains the evader, there is a detection. Otherwise there is not.

The following information describes the capabilities and constraints of a searching unit:

V = Search speed,

T = Searching time,

W = Sweepwidth (POD of the evader = 1.0 within W, zero otherwise), and

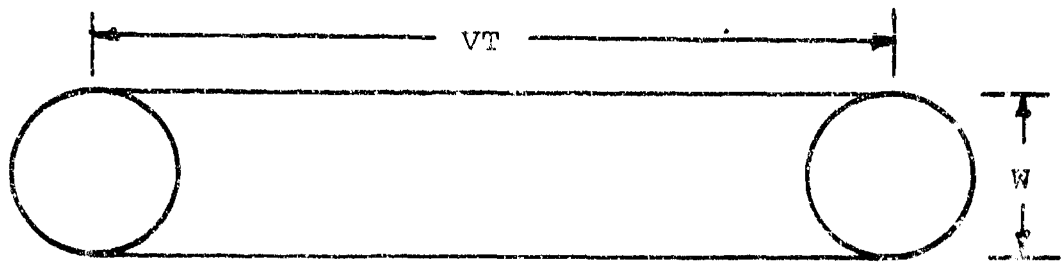
a = Area searched = $WVT + \frac{\pi W^2}{4}$ (in time T).

In Figure 5 we see that the area swept (searched) in a time interval T by the searcher, consists of displacing the initial sweepwidth circle a distance VT. If the search time, T, is sufficiently large such that $WVT \gg \frac{W^2 \pi}{4}$, then the area, a, can be approximated by:

$$a \approx WVT. \tag{12}$$

We have assumed that the evader is operating at a specific location, such as a field headquarters, within an area, A. He will remain at this location, and cannot move to avoid the searcher. We continue to invoke the principle of

randomness. This means that the searcher does not know where to search for the evader nor does the evader know where



W = Sweepwidth
V = Searcher's Speed
T = Time Searching

Figure 5. Total Area Searched

the searching units will look for him. Further, we assume that the evader is equally likely to be anywhere within the area when the search is initiated and the searcher is equally likely to begin searching at any point within the area. The searcher then begins the search with the objective of detecting the evader.

The measure of effectiveness of the searcher is the probability of detection, POD. The searcher would like to exhaustively cover the whole area and if he did the POD would be 1.0. Otherwise the POD is the ratio of the area searched, a, to the total area A, or

$$POD|A = \frac{a}{A}. \tag{13}$$

Figure 6 shows a rectangular area, A = hS, which may contain the evader and the area a covered by a searching unit.

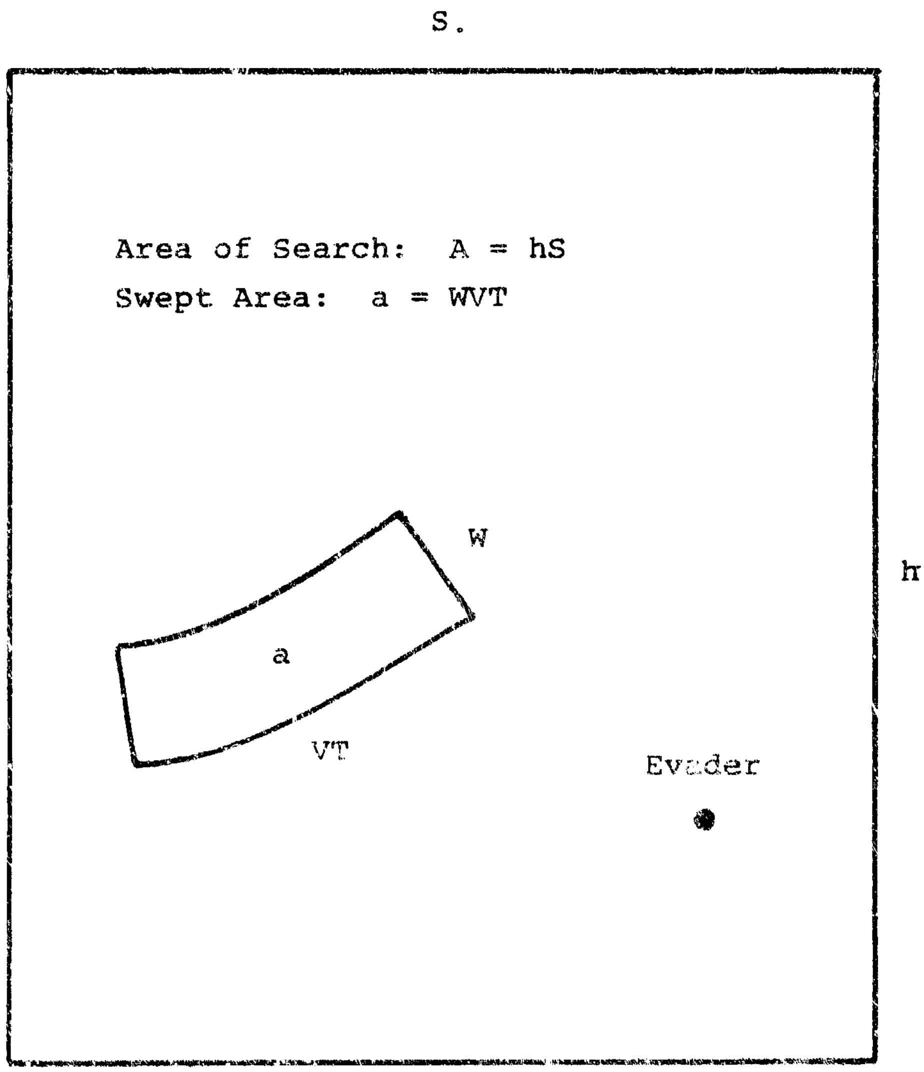


Figure 6. Rectangular Search Area

An area searching problem involves two dimensions. Under some circumstances, we can fix one dimension of the rectangle and only have the other one variable. For example, if we were searching for an airplane that flew over water along a track ten miles from the coast, we may search out to sea as far as twenty miles along some length of coastline. Thus, we have fixed one side of our search rectangle but the other dimension, along the Coast, would still be chosen to reflect realistic bounds on the problem. We now substitute the total search area hS, for A, into Equation (13), which yields:

$$POD|A = \frac{a}{hS} \tag{14}$$

The term $\frac{1}{h}$ is a constant. Now, if we compare Equation (14) with Equation (4), for the one dimensional border search problem we see that the mathematical form is the same.

Now, we will expand the problem to two searchers. Two searchers, Unit 1 and Unit 2, with different sweepwidths, speeds and search times will sweep areas a_1 and a_2 respectively. Here,

$$a_i = W_i V_i t_i, \qquad i = 1,2. \tag{15}$$

The two swept areas a_1 and a_2 are not allowed to overlap, and of course must be within the overall search area, A. The search area, A, is partitioned into two parts, one of size

A_1 searched by Unit 1 and the other of size A_2 searched by Unit 2, so that

$$A = A_1 + A_2. \tag{16}$$

It is assumed that the evader is equally likely to be at any point in the search area. Therefore, the total probability of detecting the evader is:

$$P(\text{Det}) = \frac{a_1 + a_2}{A_1 + A_2} = \frac{a}{A}, \tag{17}$$

where:

$a_1 + a_2 = a$, the total area swept out by both search units.

The conclusion in the area search is analogous to the border search; the probability of detection is not affected by the partitioning of the search area.

We have found for this problem that there is no way to partition the search area and improve the probability of detection. The POD can be degraded[1] but not improved. In the next section, we shall model the search as a game and exploit the fact that the searcher and the evader have conflicting goals.

---

[1]By overlapping search areas.

III. MODELING THE SEARCH AS A GAME

Up to now, the information that we have obtained is that the probability of detecting the evader is $(a_1+a_2)/A$ regardless[1] of how the searcher chooses to allocate the two searching units. We will now expand the problem to allow the searcher and the evader alternative courses of action. The partitioning of A will define a choice for the evader. He must choose to locate in either A_1 or A_2, and thus the evader has only these two choices, to operate (hide) in A_1 or A_2. The partitioning of the search area also provides the searcher with alternative courses of action on how he allocates his resources.

The theory of games is useful in analyzing conflicts between opposing parties [Refs. 3,4]. The two players in this game are the searcher and the evader. The conflict is that the searcher wants the probability of detecting the evader to be high while the evader wants it to be low.

A game matrix is constructed from the outcomes of all possible courses of action of the two players. We have assumed that the searcher partitions the search area, A, into two pieces A_1 and A_2. We will further assume that this partitioning is known to the evader. This does not impart any information to the evader about the searcher's course of

---

[1] Providing there is no overlap of search areas.

action, i.e., the area in which he will search. The only restriction on the partitioning of A is that each of the two pieces, A_1 and A_2, be at least as large as the sum of the search capabilities, a_1 and a_2, of the two searching units. The searcher has several ways that he can allocate his two search units to the operating areas. The four alternatives of the searcher that involve both searching units are presented in Table 1 below. The entries in the matrix identify which search units operate in each of the two areas, A_1 and A_2.

TABLE 1. POSSIBLE STRATEGIES OF THE SEARCHER

Searcher Strategy	A_1	A_2
S_1	Unit 1	Unit 2
S_2	Unit 2	Unit 1
S_3	Units 1 & 2	None
S_4	None	Units 1 & 2

For completeness, the evader's strategies are shown in Table 2.

TABLE 2. Possible Strategies of the Evader

Evader Strategy		
E_1	Locate in A_1	(Not A_2)
E_2	Locate in A_2	(Not A_1)

This situation may be treated as a two-person zero-sum game characterized by a payoff matrix which specifies the results of all possible combinations of the strategies of both players. In our case, the entries in the payoff matrix are the probabilities of detection of the evader by the searcher. For example, if the evader hid in A_2 (Strategy 2) and the searcher chose his strategy S_1 (where Unit 2 searches A_2 and Unit 1 searches in A_1), the probability of detecting the evader would be a_2/A_1. If the evader chose his strategy E_2 and the searcher chose his strategy S_4, the probability of detection would be $(a_1+a_2)/A_2$. In game theory, it is conventional to present the payoff matrix with favorable payoff to the row player. The probability of detection is a measure of success of the searcher. Therefore, the searcher will be the row player and the evader will be the column player. The payoff matrix for the game is presented below in Table 3. It is useful to establish a hierarchy of the payoff quantities.

TABLE 3. PAYOFF MATRIX FOR TWO SEARCHERS WITH AREA PARTITIONED

		EVADERS' STRATEGIES	
		E_1	E_2
Searchers Strategies	S_1	$\frac{a_1}{A_1}$	$\frac{a_2}{A_2}$
	S_2	$\frac{a_2}{A_1}$	$\frac{a_1}{A_2}$
	S_3	$\frac{a_1+a_2}{A_1}$	0
	S_4	0	$\frac{a_1+a_2}{A_2}$

We will call the more capable search unit, Unit 1. This means that Unit 1 sweeps more area than Unit 2 during the search, i.e., $a_1 > a_2$. We will call the larger of the two search areas A_1, i.e., $A_1 > A_2$. We lose no generality with these assumptions.

We will examine the case when $a_1/A_1 > a_2/A_2$ in detail. The second case, when $a_1/A_1 < a_2/A_2$ may be analyzed similarly. It is convenient to replace the entries in payoff matrix (Table 3) with the symbols in Table 4 below. This will be useful later when we analyze the game.

TABLE 4. SYMBOLIC PAYOFF MATRIX

	E_1	E_2
S_1	α	β
S_2	γ	δ
S_3	$\gamma + \gamma$	0
S_4	0	$\beta + \delta$

e.g., $\beta = \dfrac{a_2}{A_2}$

Since $\dfrac{a_1}{A_1} > \dfrac{a_2}{A_2}$; $\alpha > \beta$.

Since $a_1 > a_2$; $\alpha > \gamma$ and $\delta > \beta$.

Since $A_1 > A_2$; $\delta > \alpha$ and $\beta > \gamma$.

This yields the following hierarchy, $\delta > \alpha > \beta > \gamma$. Also since $\delta > \alpha$ and $\beta > \gamma$; $\beta + \delta > \alpha + \gamma$.

Therefore,

$$\frac{a_1}{A_2} > \frac{a_1}{A_1} > \frac{a_2}{A_2} > \frac{a_2}{A_1}$$

and

$$\frac{a_1+a_2}{A_2} > \frac{a_1+a_2}{A_1}.$$

This still leaves the relationship between a_1/A_1 and $(a_1+a_2)/A_2$ unknown, but as we will see later that it does not affect the outcome of the evaluation of the game.

We will now consider two examples. The dimensions would be chosen to be the same (e.g., sq. kilometers). They will cancel out of the equations for the probability of detection. The important point in the examples that follow are the relative sizes of the area to be searched and the available ability to search.

Example 1. Two search units searching for one evader where,

$$A = 100, \quad A_1 = 60, \quad \text{and} \quad A_2 = 40;$$

$$a = 20, \quad a_1 = 15, \quad \text{and} \quad a_2 = 5.$$

Note that

$$a_1 > a_2, \quad A_1 > A_2, \quad \text{and} \quad \frac{a_1}{A_1} > \frac{a_2}{A_2}.$$

Substituting these values into the payoff matrix yields:

	E_1	E_2
S_1	$\frac{15}{60}$	$\frac{5}{40}$
S_2	$\frac{5}{60}$	$\frac{15}{40}$
S_3	$\frac{5+15}{60}$	0
S_4	0	$\frac{5+15}{40}$

Note that $\delta > \alpha+\gamma$ in the matrix, i.e., $\frac{15}{40} > \frac{20}{60}$. It is more convenient to deal with a matrix of integers than fractions. Therefore we multiply each entry by 24. This scales the value of the game by a factor of 24 and the following game matrix results.

	E_1	E_2
S_1	6	3
S_2	2	9
S_3	8	0
S_4	0	12

Example 2. Two search units searching for one evader where,

$$A = 270, \quad A_1 = 150, \quad \text{and} \quad A_2 = 120;$$

$$a = 100, \quad a_1 = 60, \quad \text{and} \quad a_2 = 40.$$

Again, note that $a_1 > a_2$, $A_1 > A_2$ and $\frac{a_1}{A_1} > \frac{a_2}{A_2}$.

Substituting into the payoff matrix yields

$$\begin{array}{ccc}
 & E_1 & E_2 \\
S_1 & \frac{60}{150} & \frac{40}{120} \\
S_2 & \frac{40}{150} & \frac{60}{120} \\
S_3 & \frac{40+60}{150} & 0 \\
S_4 & 0 & \frac{40+60}{120}
\end{array}$$

We see here that $\delta < \alpha+\gamma$, i.e., $\frac{60}{120} < \frac{100}{150}$. The hierarchy of δ and $\alpha+\gamma$ are reversed in these two examples. When we solve the games, we will see that it does not affect the outcome since the order of all of the other parameters has been established.

We multiply all of the entries in the matrix by 30 to obtain the following payoff matrix.

	E_1	E_2
S_1	12	10
S_2	8	15
S_3	20	0
S_4	0	25

We will evaluate the game according to the minimax theorem. This dictates the manner in which both the searcher and the evader must choose their strategies and the value of the game that results from these choices. The details of evaluating a game subject to the conditions of the minimax theorem are presented in Appendix A.

At this point, we may solve the 4x2 game in Tables 3 and 4. An analytic solution is preferred but is not possible in an Mx2 game (where M > 2). We can solve the underlying 2x2 games analytically. For the Mx2 game, we would resort to a graphical solution. This requires that the entries in the payoff matrix are specified numerically or at least the hierarchy of the elements is established. The examples are plotted in Figure as a function of the evader's strategies. For either game, we can see in Figure 7 that the four strategies of the searcher all intersect at a single point. The

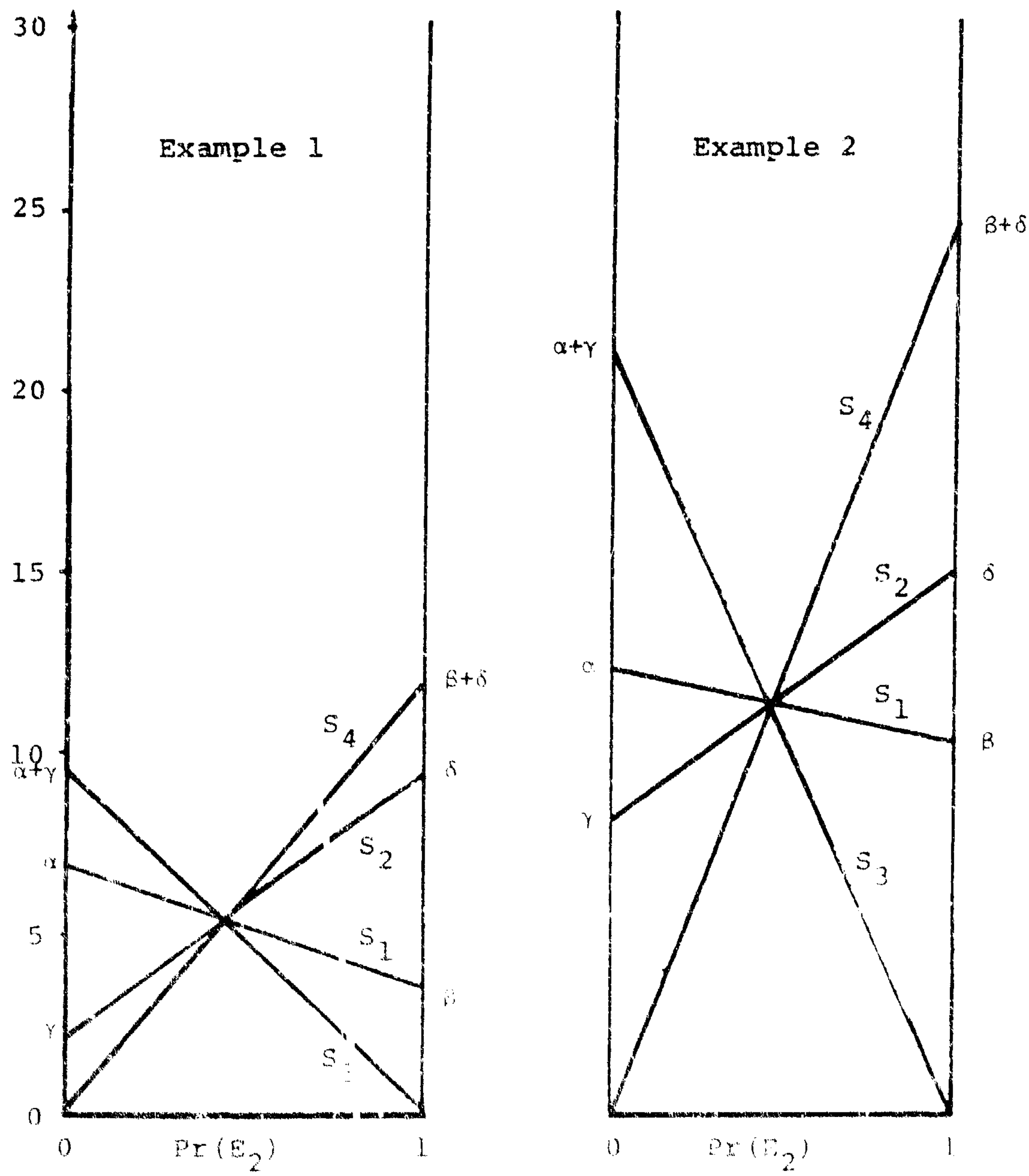


Figure 7. Examples of 4 X 2 Games

general 4x2 game would have six intersections of the strategies (4 things taken 2 at a time). There are only two independent strategies, defined by α, β, γ and δ. The other two strategies can be constructed from the first two by elementary matrix operations. This is shown in Appendix C.

The intersection of all four strategies of the searcher are at the same point. This is the value of the game which is $(a_1+a_2)/(A_1+A_2)$, as expected.

The probability of detection computed as the value of the game is the same as computed previously by taking the ratio of areas. The searcher could choose any of his four pairs of strategies 1-2, 1-4, 2-3 and 3-4. If he plays the strategies optimally, he would achieve a probability of detection equal to $(a_1+a_2)/(A_1+A_2)$.

All six subgames are analyzed in Appendix B. Two of the games have saddlepoints. The optimal strategies of the searcher and evader are computed for the remaining four games. Any single search is a Bernoulli trial which results in either a success or a failure. When both players choose their strategies optimally, the expected value of the probability of detection is the value of the game.

It is shown in Appendix B that the value of the games are the same. We can see from the grpahical display of the game, Figure 7, that some pairs of strategies range over more extreme values than others. Compare the pair of Strategies 1 and 2 with 3 and 4. We see that there is more potential for

fluctuation of the results in 3 and 4 where the values may possibly range from zero to $\beta+\delta$. The game with Strategies 1 and 2, on the other hand, has a smaller range of possible values, from γ to δ.

We have four 2x2 games with the same value. In accordance with the expectation-variance principle of choice for decisions, the choice of which set of strategies to use for searching should be based on the minimum variance. By choosing the alternatives providing minimum payoff variance, the risk of achieving a probability of detection substantially less than the value of the game is minimized. A similar argument has been advanced about investment portfolios [Ref. 5]. If all portfolios have the same expected return on investment, the portfolio with the largest variance is the one with the greatest risk. Conversely the portfolio with the least variance has the least risk.

If we look at the combinations of strategies in Figure 7, we can see that searchers' Strategies 1 and 2 vary the least with respect to the expected value. Therefore, we may select the alternative with minimum variance by inspection. The computations presented in Table 5 confirm this.

One result of modeling the search as a game is that it allows us to compute the variance as well as the expected value of the search strategies. The variance computation of a 2x2 game is described in Appendix A. The variance and standard deviation for each of the 2x2 subgames in the two examples is shown in Table 5.

TABLE 5. VARIANCE AND STANDARD DEVIATION FOR EACH STRATEGY PAIR

	Example 1		Example 2	
Strategy Pair	Variance	Std. Dev.	Variance	Std. Dev.
1-2	.00875	.094	.003841	.062
1-4	.015	.122	.01372	.117
2-3	.02333	.153	.03841	.196
3-4	.0400	.200	.13717	.370
Value of Game	.20		.370	

A. EXTENSION TO MORE SEARCH AREAS

The game approach to searching can be applied to problems involving three or more search areas and searching units. At this point the combinations of ways to search become very large. For example, if there are two search units and three areas to search, we can count nine ways to put the two searchers in the three areas and thus nine pure strategies for the searcher. This would lead to $\binom{9}{3} = 84$ 3x3 games to analyze.

One simple example is where one searching unit looks for one evader in one of three different regions. This is similar to trying to pick the walnut shell that has the pea hidden under it. Of course if you pick the right shell (search area)

the probability of detection is 1.0. The payoff matrix for this search model is shown below.

	E_1	E_2	E_3
S_1	P_{11}	0	0
S_2	0	P_{22}	0
S_3	0	0	P_{33}

Using the method in Epstein [Ref. 6] for finding optimal strategies of NxN games, we find

$$P(E_1) = P(S_1) = \frac{P_{22} P_{33}}{P_{11} P_{22} + P_{22} P_{33} + P_{33} P_{11}},$$

$$P(E_2) = P(S_2) = \frac{P_{11} P_{33}}{P_{11} P_{22} + P_{22} P_{33} + P_{33} P_{11}},$$

$$P(E_3) = P(S_3) = \frac{P_{11} P_{22}}{P_{11} P_{22} + P_{22} P_{33} + P_{33} P_{11}},$$

and $$V = \frac{P_{11} P_{22} P_{33}}{P_{11} P_{22} + P_{22} P_{23} + P_{33} P_{11}}.$$

If $P_{11} = P_{22} = P_{33} = P$, and

$$P(S_i) = P(E_j) = 1/3, \quad i = 1,2,3, \quad j = 1,2,3,$$

then the value of the game = $P/3$, the variance = $P^2/3$, and the standard deviation = $P/\sqrt{3}$.

The searcher and evader would play each of their strategies with probability 1/3 and achieve a probability of detection (payoff) equal to 1/3 of what it would be in any of the three areas.

If P = 1 (the shell game) then

$$\text{Probability of Finding Pea} = 1/3 \ ,$$

$$\text{Variance of Probability} = 1/3 \ ,$$

$$\text{Standard Deviation} = 1/\sqrt{3} \ , \text{ and}$$

$$\text{Value of Game} \pm \text{Standard Deviation} = 1/3 \pm 1/\sqrt{3}.$$

We have analyzed search problems involving stationary evaders using game theory. It is possible to expand beyond 2x2 games but the potential combinations of strategies grow very quickly. Next we will address searching for a moving target which we will model with random search. We will continue to use game theory to analyze the problem.

IV. RANDOM SEARCH

The concept of random search was introduced by B. Koopman in 1946 [Ref. 1]. In random search, the evader may move into an area already covered by the searcher. Therefore, even if the searcher covers the area exhaustively, he may not find the evader. By repeatedly covering the area, the POD then goes asymptotically to 1.0. This model for the POD is the exponential

$$POD = 1 - e^{-a/A},$$

where:

a = Area swept by search unit, and

A = Area in which evader is operating.

The POD always increases with increasing coverage in the random search model. Random search always gives a smaller value for the probability of detection than the ratio of areas model, as shown in Table 6.

The next step in the development of the search problem is to allow random search in our previously developed examples. The evader is now free to move within the partitioned area that he has chosen, but is not allowed to cross the partition between A_1 and A_2. We continue to model the problem with the four searcher strategies previously described. The entries

TABLE 6. DETECTION PROBABILITY COMPARISON

a/A	Random Search	Ratio of Areas
.25	.2212	.2500
.5	.3935	.5000
1.0	.6321	1.0000
2.0[1]	.8647	1.0000

in the payoff matrix are now the probabilities of detection using random search. The arguments of the POD are the a_i and A_j shown previously. The general form of the 4x2 game matrix is shown below.

$$
\begin{array}{ccc}
 & E_1 & E_2 \\
S_1 & 1-e^{-a_1/A_1} & 1-e^{-a_2/A_2} \\
S_2 & 1-e^{-a_2/A_1} & 1-e^{-a_1/A_2} \\
S_3 & 1-e^{-(a_1+a_2)/A_1} & 0 \\
S_4 & 0 & 1-e^{-(a_1+a_2)/A_2}
\end{array}
$$

[1]In the ratio of areas model, once the area is covered, the POD = 1.0. Additional coverage is redundant.

Example 1. When we use the previous values, $a_1 = 15$, $a_2 = 5$, $A_1 = 60$ and $A_2 = 40$, we obtain:

	E_1	E_2		E_1	E_2
S_1	$1-e^{-1/4}$	$1-e^{-1/8}$	S_1	.22120	.11750
S_2	$1-e^{-1/2}$	$1-e^{-3/8}$	S_2	.07996	.31271
S_3	$1-e^{-1/3}$	0	S_3	.28347	0
S_4	0	$1-e^{-1/2}$	S_4	0	.39347

Example 2. Similarly, we use the values $a_1 = 60$, $a_2 = 40$, $A_1 = 150$, and $A_2 = 120$ to obtain:

	E_1	E_2		E_1	E_2
S_1	$1-e^{-2/5}$	$1-e^{-1/3}$	S_1	.32968	.28347
S_2	$1-e^{-4/15}$	$1-e^{-1/2}$	S_2	.23407	.39347
S_3	$1-e^{-2/3}$	0	S_3	.48658	0
S_4	0	$1-e^{-5/6}$	S_4	0	.56540

It is useful to consider some other examples using random search where the coverage factor is greater than 1.0. The same four searcher strategies will still be used while the parameters a_1, a_2, A_1 and A_2 will be chosen to give coverage factors greater than 1.0.

<u>Example 3</u>. $A = 500;\ A_1 = 400;\ A_2 = 100$

$$a = 300;\ a_1 = 200;\ a_2 = 100$$

$$\frac{a_1}{A_1} = \frac{1}{2},\quad \frac{a_2}{A_2} = 1,\quad \frac{a_2}{A_1} = \frac{1}{4},\quad \frac{a_1}{A_2} = 2$$

	E_1	E_2		E_1	E_2
S_1	$1-e^{-1/2}$	$1-e^{-1}$	S_1	.39347	.63212
S_2	$1-e^{-1/4}$	$1-e^{-2}$	S_2	.22120	.86466
S_3	$1-e^{-3/4}$	0	S_3	.52763	0
S_4	0	$1-e^{-3}$	S_4	0	.95021

Note here the strong coverage of A_2.

<u>Example 4</u>. $A = 500;\ A_1 = 350;\ A_2 = 150$

$$a = 200;\ a_1 = 150;\ a_2 = 50$$

$$\frac{a_1}{A_1} = \frac{3}{7}; \quad \frac{a_2}{A_2} = \frac{1}{3}; \quad \frac{a_2}{A_1} = \frac{1}{7}; \quad \frac{a_1}{A_2} = 1$$

	E_1	E_2
S_1	$1-e^{-3/7}$	$1-e^{-1/3}$
S_2	$1-e^{-1/7}$	$1-e^{-1}$
S_3	$1-e^{-4/7}$	0
S_4	0	$1-e^{-4/3}$

	E_1	E_2
S_1	.34856	.28347
S_2	.13312	.63212
S_3	.43528	0
S_4	0	.73640

We next address all the 2x2 subgames in each example and compute the value of the game, the variance and the standard deviation. In Example 3, the S_1 S_4 game now has a saddle-point since the hierarchy of the elements within the payoff matrix have changed. The data is presented in Table 7.

We see that the random search results are different than the area search results. The expected value of the probability of detection in Examples 1 and 2 is lower as expected. According to game theory, we would choose the pair of search strategies which yield the largest expected value. In all cases this is the S_1-S_2 pair. Further, we can see that this pair of strategies has the smallest variance. We also see that the diminishing returns effect of random search produces lower expected payoff in the more extreme search patterns.

TABLE 7: EXPECTED VALUE AND STANDARD DEVIATION FOR RANDOM SEARCH EXAMPLES

	Example 1		Example 2		Example 3		Example 4	
	A = 100; a = 20		A = 270; a = 100		A = 500; a = 300		A = 500; a = 200	
	A = 60; a = 15		A = 150; a = 60		A = 400; a = 200		A = 350; a = 150	
	A = 40; a = 5		A = 120; a = 40		A = 100; a = 100		A = 150; a = 50	
Strategy Pair	Val.	Std. Dev.	Val.	Std. Dev.	Val.	Std. Dev.	Val.	Std. Dev.
S_1-S_2	.11767	.07667	.30819	.04281	.43525	.13486	.32370	.08756
S_1-S_4	.17506	.10038	.30477	.08057			.32025	.10853
S_2-S_3	.17172	.12552	.29638	.13589	.38957	.25611	.29450	.21801
S_3-S_4	.16477	.16477	.26152	.26152	.33925	.33925	.27357	.27357

V. SUMMARY

The game theory approach to modeling search is useful when one wants to gain more information about the problem than the probability of detection. If each alternative yields the same expected value, the choice among alternatives would be based upon minimum variance. The alternative that yields the largest expected values would be chosen by the searcher, if the expected values were all different. In some particular applications it may even be desirable to select the alternative with the largest variance or some function of the expected value and variance. The user can learn more about the potential outcomes of his problem by examining the results of the game model. He now has the variance available to further describe the outcome. He would use the area ratio or random search model (whichever was applicable) for the probability of detection.

An interesting area for future work is relating the expected value predicted by game theory to the probability of detection determined by experiment, since experiments that are expensive or physical situations that occur infrequently don't allow for many repetitions.

In a game situation the outcome is determined by the strategies chosen by each player. It would be interesting to know the relationship between the outcome of the game and the variance and the probability distributions of each player

and their variances, and the number of times that the game is played.

It is hoped that this approach to search using game theory is useful to those in the operations research community and that it may stimulate further work in the area.

APPENDIX A

GAME THEORY CALCULATIONS

In this appendix, we will develop the optimal probabilities for the searcher and evader when they play the game according to the minimax theorem. We will also compute the value of the game when each player uses these optimal probabilities to select his strategy.

The payoff matrix by convention is the payoff of the column player to the row player. Since the probability of detection is a positive payoff to the searcher, we will make the searcher the row player and the evader the column player. Another common convention is to call the row player Blue and the column player Red. We will use the terminology of the problem, searcher and evader.

A zero sum game represents an exchange between two players. Whatever the column player considers to be detrimental, the row player considers to be beneficial and vice versa. If the payoff matrix represents a physical exchange (of money, for example) the term zero-sum is clear. In our case the probability of detection of the evader may be considered a loss to the evader and a gain to the searcher.

The searcher and evader play the game according to the minimax theorem. This says that the row player, the searcher, selects his strategies to maximize the minimum expected payoff. The column player, the evader, selects his strategies

to minimize the maximum expected loss. These two expected values are called the lower and upper value of the game.

The mix of strategies that maximizes the minimum expected gain for the row player is called the optimal mix of strategies. This assures that he will gain at least the lower value of the game. A similar argument for the column player says that his optimal mix strategies would assure him of a loss no greater than the upper value of the game. One of the elegant features of game theory is that when both players select their strategies optimally that the upper and lower value of the game are equal. This is then simply called the value of the game. The value of the game is the expected value of the probability of detection when the searcher and the evader randomly select from their optimal strategies.

A game with a saddlepoint occurs when the minimum of the column maximums is equal to the maximum of the row minimums. Games with saddlepoints are played in only one way. The row player and column player each select the same strategy for every play of the game. The payoff will be the same after every play. A game should be checked for a saddlepoint since the optimal strategies and the value of the game formulas (that we will develop next) do not apply if a saddlepoint exists. The optimal strategies of the searcher and evader are developed as follows. Consider the 2x2 games (Eq. A-1) below:

$$
\begin{array}{llll}
 & & \text{Evader} & \\
 & & E_1 & E_2 \\
\text{Searcher} & S_1 & P_{11} & P_{12} \\
 & S_2 & P_{21} & P_{22}
\end{array}
\tag{A-1}
$$

where:

E_1 and E_2 are the evaders' strategies;

S_1 and S_2 are the searchers' strategies; and

P_{ij} is the payoff by the evader to the searcher when the searcher chooses strategy i and the evader chooses strategy j.

If no saddlepoint exists, the two largest elements in a 2x2 game will be a diagonal of the payoff matrix.

Let $P_{11} \geq P_{22} > P_{12} \geq P_{21}$. The payoff may be graphed (Figure 8) as a function of the mix of strategies. The maximum expected loss by the evader is shown by the darkened upper line segments. Since the evader plays to minimize the maximum expected payoff, he would play at the point where the two upper lines intersect. This means that he would play strategies E_1 and E_2 with probabilities 1-Y and Y as shown in Figure 8.

To solve for Y we equate both lines at the intersection of the two straight lines which yields:

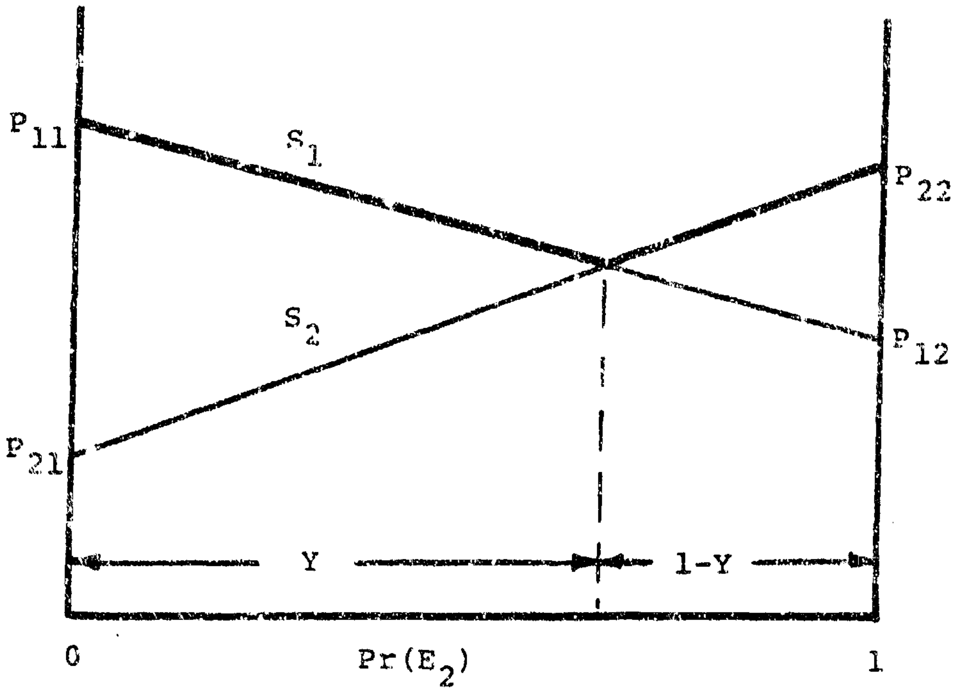


Figure 8. Graphically Solving for the Evader's Strategies

$$P_{21} + (P_{22} - P_{21})Y = P_{11} - (P_{11} - P_{12})Y \tag{A-2}$$

Solving for Y and 1-Y yields:

$$Y = \frac{P_{11} - P_{21}}{P_{11} - P_{12} - P_{21} + P_{22}} ,$$

$$1-Y = \frac{P_{22} - P_{12}}{P_{11} - P_{12} - P_{21} + P_{22}} . \tag{A-3}$$

The evader will now select his strategy randomly from E_1 and E_2 with the probabilities 1-Y and Y, respectively.

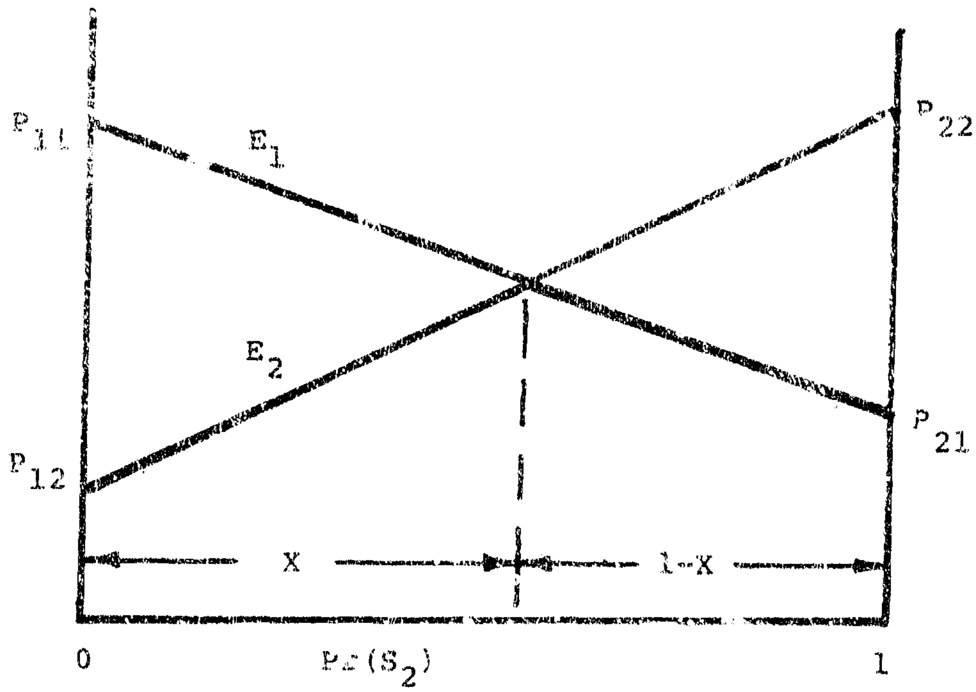


Figure 9. Graphically Solving for the Searcher's Strategies

The solution for the searchers' probabilities distribution is analogous. The payoff is plotted as a function of the percent of the searchers' strategies in Figure 9 above. Since the searcher is playing to maximize the minimum payoff, he is solving the graph from the bottom up (vice the evaders' top down). The searcher operates at the intersection of the two lines. This is the maximum of the minimum expected payoff. We solve for the probabilities of the searchers' strategies by equating the two straight lines at the intersection which yields:

$$X = \frac{P_{11} - P_{12}}{P_{11} - P_{12} - P_{21} + P_{22}},$$

$$1-X = \frac{P_{22} - P_{21}}{P_{11} - P_{12} - P_{21} + P_{22}} \qquad \text{(A-4)}$$

where:

$1-X$ = Probability of playing S_1, and

X = Probability of playing S_2.

The value of the game, V, is the expected value which is determined from the following equation.

$$V = (1-X)(1-Y)P_{11} + (1-X)YP_{12} + X(1-Y)P_{21} + XYP_{22} \qquad \text{(A-5)}$$

This reduces to:

$$V = \frac{P_{11}P_{22} - P_{12}P_{21}}{P_{11} - P_{12} - P_{21} + P_{22}} \qquad \text{(A-6)}$$

The payoff matrix fully describes a game. The value of the game and the probabilities of each strategy are a function of the payoff matrix.

Variance of a Game

convenient computation form for the variance of a quantity X is

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

We simply compute each of the quantities on the right hand side of the equation and subtract. The game G is described by the payoff matrix P where:

$$P = \begin{array}{ccc} & E_1 & E_2 \\ S_1 & P_{11} & P_{12} \\ S_2 & P_{21} & P_{22} \end{array}$$

and S_1, S_2, E_1 and E_2 are the optimal strategies of the searcher and evader.

$$E(G^2) = S_1E_1P_{11}^2 + S_1E_2P_{12}^2 + S_2E_1P_{21}^2 + S_2E_2P_{22}^2$$

$$E(G)^2 = (S_1E_1P_{11} + S_1E_2P_{12} + S_2E_1P_{21} + S_2E_2P_{22})^2$$

The variance then is the difference between these two quantities, i.e.,

$$\mathrm{Var}(G) = E(G^2) - E(G)^2$$

APPENDIX B

EVALUATION OF GAMES

The 4x2 game under consideration is described by the following payoff matrix:

$$\begin{array}{ccc}
 & E_1 & E_2 \\
S_1 & \frac{a_1}{A_1} & \frac{a_2}{A_2} \\
S_2 & \frac{a_2}{A_1} & \frac{a_1}{A_2} \\
S_3 & \frac{a_1+a_2}{A_1} & 0 \\
S_4 & 0 & \frac{a_1+a_2}{A_2}
\end{array}$$

We impose the following conditions:

$$a_1 > a_2, \quad A_1 > A_2 \quad \text{and} \quad \frac{a_1}{A_1} > \frac{a_2}{A_2}$$

and make the following substitutions:

$$\alpha = \frac{a_1}{A_1}, \quad \beta = \frac{a_2}{A_2}, \quad \gamma = \frac{a_2}{A_1} \quad \text{and} \quad \delta = \frac{a_1}{A_2}$$

resulting in the hierarchy $\delta > \alpha > \beta > \gamma$, and payoff matrix,

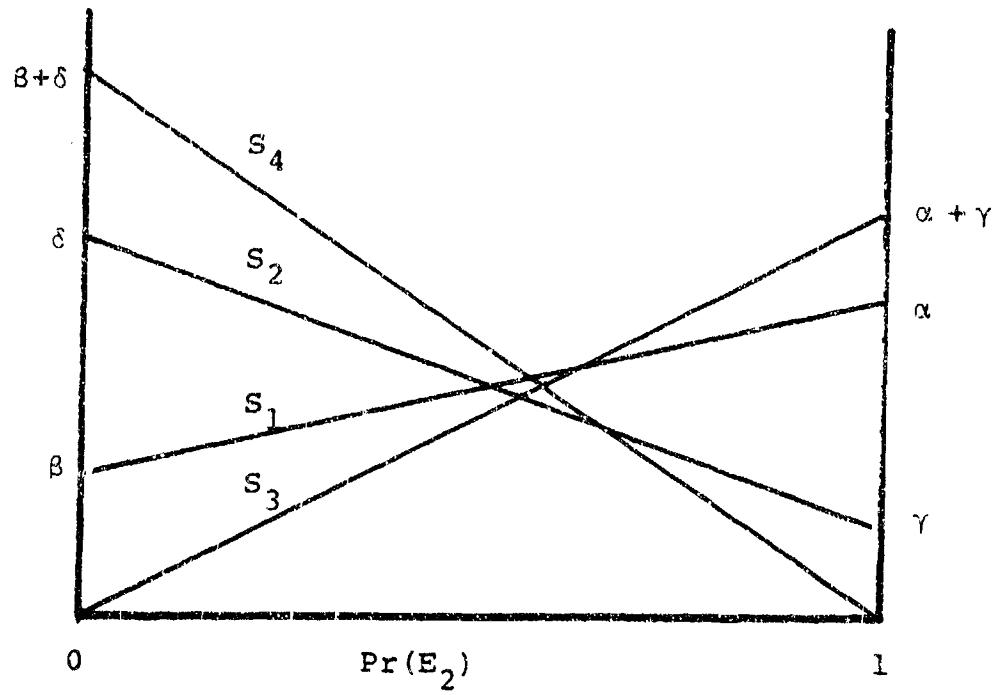


Figure 10. General Plot of 4x2 Game

	E_1	E_2
S_1	α	β
S_2	γ	δ
S_3	$\alpha+\gamma$	0
S_4	0	$\beta+\delta$

It is more convenient to work with this general matrix.

We now arbitrarily choose α, β, γ and δ according to the hierarchy and plot the game as a function of the evader's strategies.

We see in Figure 10 that there are six intersections of the four strategies. This is the general form of the 4x2 game. This would indicate that the evader would solve the game from the top down and choose strategies S_1 and S_4. But because of the partitioning of the search area and searching effort and the definition of α, β, γ and δ, this is not strictly correct. We will not solve all possible 2x2 games and show that the lines all intersect at a common point because of the definition of α, β, γ and δ.

We will use the paired notation ij to indicate which pair of the searcher's strategies we are considering. All possible 2x2 games are shown below. We find the row minimum and column maximum and check for saddlepoints.

		E_1	E_2	
12	S_1	α	β	β
	S_2	γ	δ	γ
		α	δ	

		E_1	E_2	
13	S_1	α	β	β*
	S_3	$\alpha+\gamma$	0	0
		$\alpha+\gamma$	β*	

		E_1	E_2	
14	S_1	α	β	β
	S_4	0	$\beta+$	0
		α	$\beta+\delta$	

		E_1	E_2	
23	S_2	γ	δ	γ
	S_3	$\alpha+\gamma$	0	0
		$\alpha+\gamma$	δ	

		E_1	E_2	
24	S_2	γ	δ	γ^*
	S_4	0	$\beta+\delta$	
		γ^*	$\beta+\delta$	

		E_1	E_2	
34	S_3	$\alpha+\gamma$	0	0
	S_4	0	$\beta+\delta$	0
		$\alpha+\gamma$	$\beta+\delta$	

We see that games 13 and 24 have saddlepoints. Next, all six games are plotted in Figure 11 as a function of the searchers strategies. The operating points (or solutions) for the searcher are shown by the heavy dots. The individual games for the evader may be culled from Figure 1 by taking all six possible pairs of searcher's strategies. For the case where $\alpha = a_1/A_1$, $\beta = a_2/A_2$, etc., we will show that the lines all intersect at the same point.

We will solve all six games for the optimal strategies of the searcher and the evader and the resulting value of the game. We will use the following notation:

V^m_{ij} = Value of the game using searchers strategies i and j, and

E^{ij}_K = Fraction of evaders strategy K played against searchers strategies i and j,

where:

i = 1 to 4, i j

j = 2 to 4, i j

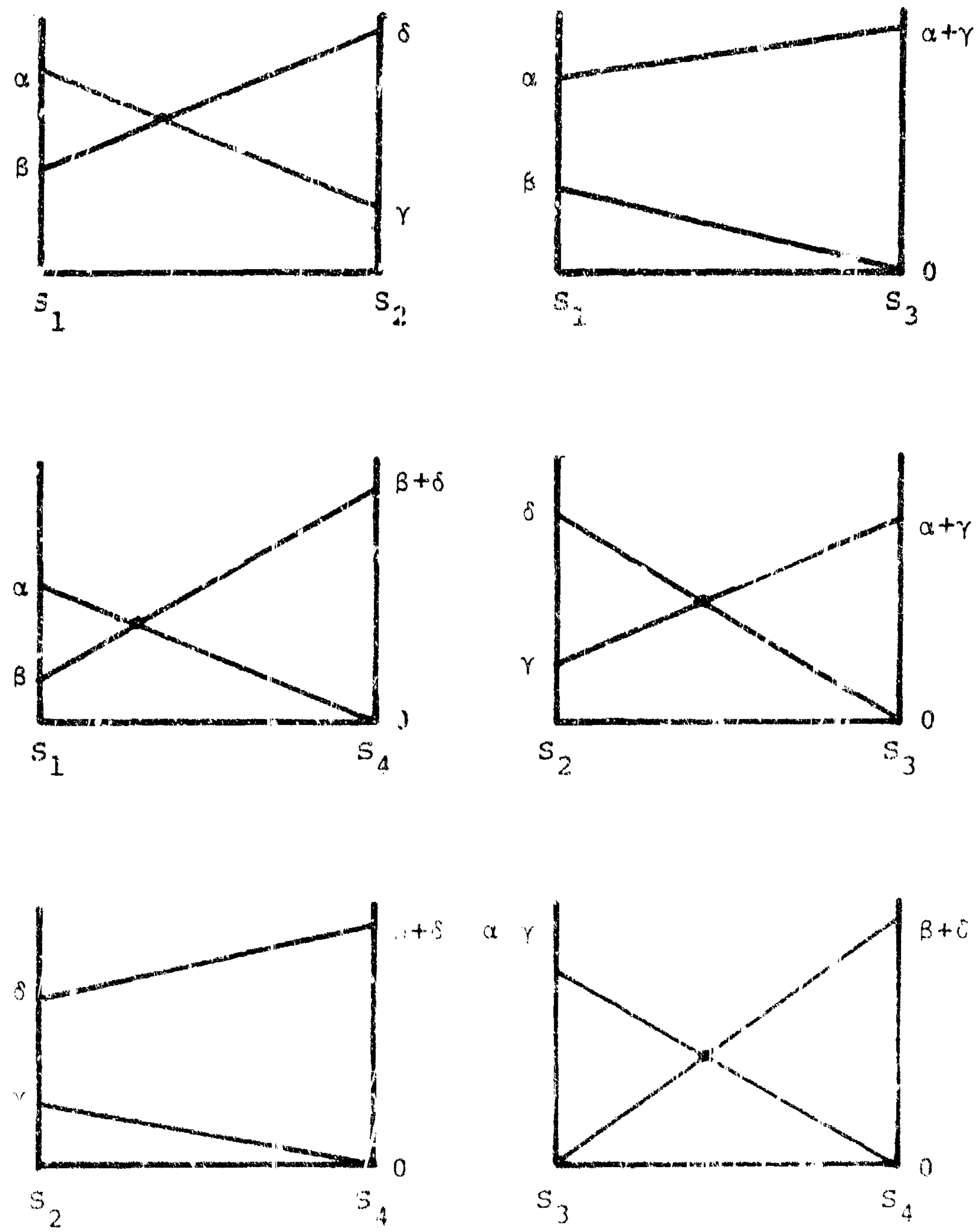


Figure 11. All Possible 2 X 2 Games

$$m = \text{Evader or Searcher},$$

$$K = 1,\ 2.$$

We will solve the value of the game for both the searcher and the evader against $S_1\ S_2$:

$$S_2 = \frac{\alpha - \beta}{\alpha - \beta - \gamma + \delta}, \qquad S_1 = \frac{-\gamma + \delta}{\alpha - \beta - \gamma + \delta},$$

$$E_1^{12} = \frac{-\beta + \delta}{\alpha - \beta - \gamma + \delta}, \qquad E_2^{12} = \frac{\alpha - \gamma}{\alpha - \beta - \gamma + \delta},$$

$$V_{12}^{E} = V_{12}^{S} = V_{14} = \frac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta}.$$

Against S_1S_3, there is a saddlepoint game with value β.

Against S_1S_4,

$$S_4 = \frac{\alpha - \beta}{\alpha + \delta}\ S_1 = \frac{\beta + \delta}{\alpha + \delta},$$

$$E_1^{14} = \frac{\delta}{\alpha + \delta}\ E_2^{14} = \frac{\alpha}{\alpha + \delta},$$

$$V_{14}^{E} = V_{14} = V_{14} = \frac{\alpha(\beta + \delta)}{\alpha + \delta}.$$

Against S_2S_3,

$$S_3 = \frac{-\gamma + \delta}{\alpha + \delta}\ S_2 = \frac{\alpha + \gamma}{\alpha + \delta},$$

$$E_1^{23} = \frac{\delta}{\alpha + \delta},\ E_2^{23} = \frac{\alpha}{\alpha + \delta},$$

$$V_{23}^{E} = V_{23}^{S} = V_{23} = \frac{(\alpha + \gamma)\delta}{\alpha + \delta}.$$

Against S_2S_4, there is a saddlepoint game with value γ. Against S_3S_4,

$$S_4 = \frac{\alpha + \delta}{\alpha + \beta + \gamma + \delta}\ S_3 = \frac{\beta + \delta}{\alpha + \beta + \gamma + \delta},$$

$$E_1^{34} = \frac{\beta + \delta}{\alpha + \beta + \gamma + \delta}\ E_2^{34} = \frac{\alpha + \gamma}{\alpha + \beta + \gamma + \delta},$$

$$V_{34}^{E} = V_{34}^{S} = V_{34} = \frac{(\alpha + \gamma)(\beta + \delta)}{\alpha + \beta + \gamma + \delta}.$$

In general, the evaders strategies E_K^{ij} are not the same for arbitrary α, β, γ and δ obeying the hierarchy. But when α, β, γ and δ are defined as a_1/A_1, etc., we see that $E_1^{12} = E_1^{14} = E_1^{23} = E_1^{34}$.

Similarly if we substitute the value of the game we see that $V_{12} = V_{14} = V_{23} = V_{34}$. This means that the game strategies all intersect at the same point, as in Figure 6. Therefore, the value of the game is $(a_1+a_2)/(A_1+A_2)$ as expected.

APPENDIX C

CONSTRUCTING STRATEGIES

There are four independent quantities that describe the search problem with two searching units and an operating area partitioned into two pieces. They are:

$$\alpha = \frac{a_1}{A_1}, \quad \beta = \frac{a_1}{A_2}, \quad \gamma = \frac{a_2}{A_1} \text{ and } \delta = \frac{a_1}{A_2}.$$

The searcher's strategies S_1 and S_2 played against the evaders strategies E_1 and E_2 yield the following payoff matrix:

$$\begin{array}{ccc} & E_1 & E_2 \\ S_1 & \alpha & \beta \\ S_2 & \gamma & \delta \end{array}$$

The searcher's strategies S_3 and S_4 played against the evader's strategies E_1 and E_2 yield the following payoff matrix:

$$\begin{array}{ccc} & E_1 & E_2 \\ S_3 & \alpha+\gamma & 0 \\ S_4 & 0 & \beta+\delta \end{array}$$

Strategies S_3 and S_4 may be expressed as a linear sum of strategies S_1 and S_2 as follows:

$$\frac{1}{\alpha\delta-\beta\gamma}\begin{matrix}\delta(\alpha+\gamma) & -\beta(\alpha+\gamma) \\ \\ -\gamma(\beta+\delta) & \alpha(\beta+\delta)\end{matrix} \quad \begin{matrix}\alpha & \beta \\ \\ \gamma & \delta\end{matrix} = \begin{matrix}\alpha+\gamma & 0 \\ \\ 0 & \beta+\delta\end{matrix} .$$

LIST OF REFERENCES

1. Operations Analysis Study Group, U.S. Naval Academy, Naval Operations Analysis, Naval Institute Press, 1977.

2. Washburn, Alan R., Class Notes on Search Theory, Naval Postgraduate School, January 1980.

3. Dresher, M., Games of Strategy Theory and Applications, Prentice Hall, 1961.

4. Williams, J.D., The Compleat Strategist, The Rand Corporation, 1954.

5. Starr, M.K. and David W. Miller, Executive Decisions and Operations Research, (Second Edition), Prentice Hall, 1969.

6. Epstein, Richard A., The Theory of Gambling and Statistical Logic, 1974.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, California 93940	2
3.	Assoc. Prof. Glenn F. Lin[illegible]y, Code 55 Department of Operations Research Naval Postgraduate School Monterey, California 93940	1
4.	Mr. Clark W. Pritchett Coast Guard Research and Development Center Groton, Conn. 06340	1
5.	Department Chairman, Code 55 Department of Operations Research Naval Postgraduate School Monterey, California 93940	1
6.	Commandant (G-OLE-1) ATTN: LT R. Lang U.S. Coast Guard Headquarters Washington D.C. 20593	1
7.	Commandant (G-OSR-3) ATTN: LT T. Hylton U.S. Coast Guard Headquarters Washington D.C. 20953	1
8.	Department of Mathematics ATTN: CAPT D. Sandell U.S. Coast Guard Academy New London, CT 06320	1
9.	Commandant (G-OP) ATTN: Mr. P. Dzura U.S. Coast Guard Headquarters Washington D.C. 20593	1
10.	USCG CHASE (WHEC 718) ATTN: CDR P. Hamilton FPO New York, NY 09566	1